PMM U.S.S.R.,Vol.45,pp.101-106
0021-8928/82/1 0101 $7.50/0

UDC 539.6

EXPANSION IN PROBLEMS OF CONTROL OF SYSTEMS WITH EQUATIONS OF THE PARABOLIC TYPE*

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The method of decomposition based on the introduction of macro-variables is applied for controlling systems with distributed parameters, which are defined by parabolic type equations in partial derivatives. Basic statements are presented for the iterative process: optimality criterion for the intermediate solution and local monotonicity with respect to the functional of the input problem.

The problem of optimal heating for a system of J infinite plates of width $l_j\ (j\in[1:J])$ is considered. The temperature of each plate $z_j(x_j, t)$ with respect to width and time satisfies the heat conduction equation

$$\partial z_j(x_j, t) / \partial t - \partial^2 z_j(x_j, t)/\partial x_j^2 = f_j(x_j,t) \in C_0\{[0, l_j]\times[0, T]\},\quad t>0,\quad 0<x_j<l_j \tag{1}$$

Initial temperature distributions are specified, and the heat exchange at the plate boundaries conforms to Newton's law

$$\lim_{t\to+0} z_j(x_j, t) = \varphi_j(x_j) \in C_0[0, l_j] \tag{2}$$

$$\partial z_j(0, t) / \partial x_j = -\beta_j^1[\eta_j^1(t) - z_j(0, t)] \tag{3}$$

$$\partial z_j(l_j, t)/\partial x_j = \beta_j^2[\eta_j^2(t) - z_j(l_j, t)]$$

$$(\beta_j^1)^2 + (\beta_j^2)^2 > 0,\quad \beta_j^1\geqslant 0,\quad \beta_j^2\geqslant 0$$

We introduce control vectors $u_j(t) = \{u_j^1(t), \ldots, u_j^I(t)\}$ which define the heating medium temperature in terms of I-dimensional vectors $b_j^k(t) = \{b_j^{1k}(t), \ldots, b_j^{Ik}(t)\}\ (k = 1, 2)$

$$\eta_j^k(t) = b_j^k(t)u_j(t),\quad k = 1, 2 \tag{4}$$

The controls are subject to the following constraints:

$$0\leqslant u_j(t)\leqslant a_j(t) \tag{5}$$

$$\sum_{j=1}^J [d_j(t)u_j(t) - c_j^1(t)z_j(0, t) - c_j^2(t)z_j(l_j, t)]\leqslant m(t) \tag{6}$$

where $d_j(t)$ are matrices of dimension $R\times I$; $c_j^k(t)$, $m(t)$ and $a_j(t)$ are, respectively, R- and I-dimensional vectors.

Constraints (5) apply to each individual plate, while (6) is the binding condition for the system as a whole. The meaning of inequality (6) can be interpreted as follows. Let the control vector components $u_j = \{u_j^1, \ldots, u_j^I\}$ for each fixed $j\in[1:J]$ define components of consumed fuel required for maintaining the heating media temperature. The system sustains losses in the course of consumption of fuel components and has inducers for maintaining the highest possible temperature at the boundary of plates and media. The difference between such losses and inducers expressed in the same measurements is positive and limited, which is specified by vector $m(t)$ in the right-hand side of (6). The components of matrices $d_j(t)$ and of vectors b_j^k, $c_j^k(t)$ $(k = 1, 2)$ have then the meaning of respective proportionality coefficients, and the concrete meaning of condition (6) is finally elucidated.

For the above formulation /of the problem/ a typical functional is of the form

$$-\sum_{j=1}^J\int_0^{l_j}[z_j(\xi_j, T) - e_j(\xi_j)]^2 d\xi_j\to\max \tag{7}$$

where $e_j(x_j)$ are given functions on segments $[0, l_j]$, $j\in[1:J]$.

It is assumed that components $b_j^{ik}(t)$, $d_j^{ir}(t)$, $c_j^{rk}(t)$, $m^r(t)$, $a_j^i(t)$ are bounded measurable functions on segment $[0, T]$, and $e_j(x_j)\in L_2[0, l_j]$. We seek control vectors $u_j(t)\in L_\infty^I[0, T]$, $j\in[1:J]$ and their respective temperature functions $z_j(x_j, t)$, $j\in[1:J]$ which maximize functional

*Prikl.Matem.Mekhan.,45,No.1,137-144,1981

(7) and satisfy constraints (4)—(6) and, also, Eqs.(1) of heat conduction together with initial and boundary conditions (2) and (3) in the sense of the theory of distribution.

The dimension of the input problem with respect to controls is $J \times I$. Reduction to problems of lower dimensions on the basis of introduction of macrocontrols proposed in /1/ is indicated subsequently. The construction is based on the principle of duality for extremal problems in Banach spaces /2/ to which the input and all intermediate problems are reduced with the use of Green's function in the method of decomposition. The validity of duality theorems for problems of control with distributed parameters of the type considered here with conditions $b_j^{ik}(t), d_j^{ir}(t), c_j^k(t) \geqslant 0$ satisfied was proved in Ter-Krikorov's paper (*).

The iterative process is constructed as follows. We introduce macrocontrols and weight functions

$$U^i(t) = \sum_{j=1}^{J} u_j^i(t), \ i \in [1:I], \quad \alpha_j^i(t) = \frac{u_j^i(t)}{U^i(t)}, \ j \in [1:J], \ i \in [1:I]$$

We fix $\alpha_j^i(t)$ and obtain a problem with macrocontrols by substituting in the input problem for (4)—(6), respectively, the relationships (8)—(10)

$$\eta_j^k(t) = \sum_{i=1}^{I} B_j^{ik}(t) U^i(t), \ B_j^{ik}(t) = b_j^{ik}(t) \alpha_j^i(t) \tag{8}$$

$$0 \leqslant \alpha_j^i(t), \quad U^i(t) \leqslant a_j^i(t) \tag{9}$$

$$\sum_{i=1}^{I} D^i(t) U^i(t) - \sum_{j=1}^{J} [c_j^1(t) z_j(0, t) + c_j^2(t) z_j(l_j, t)] \leqslant m(t), \quad D^i(t) = \sum_{j=1}^{J} d_j(t) \alpha_j^i(t) \tag{10}$$

For the problem with macrocontrols we consider its conjugate /3/

$$-\partial w_j(x_j, t) / \partial t - \partial^2 w_j(x_j, t) / \partial x_j^2 = 0, \quad \lim_{t \to T-0} w_j(x_j, t) = -2[z_j(x_j, T) - e_j(x_j)] \tag{11}$$

$$\partial w_j(0, t) / \partial x_j = -\gamma_j^1(t) + \beta_j^1 w_j(0, t) \quad \partial w_j(l_j, t) / \partial x_j = \gamma_j^2(t) - \beta_j^2 w_j(l_j, t)$$

$$\gamma_j^k(t) = \delta(t) c_j^k(t); \quad k = 1, 2, j \in [1:J]$$

$$\delta(t) D^i(t) + \sum_{j=1}^{J} [-\beta_j^1 w_j(0, t) B_j^{i1}(t) - \beta_j^2 w_j(l_j, t) B_j^{i2}(t) + \alpha_j^i(t) \mu_j^i(t)] \geqslant 0, \quad i \in [1:I]$$

$$\delta(t) \geqslant 0, \quad \mu_j(t) \geqslant 0, \quad j \in [1:J]$$

$$\sum_{j=1}^{J} \left\{ \int_0^T \mu_j(\tau) a_j(\tau)\, d\tau + \int_0^{l_j} w_j(\xi_j, 0) \varphi_j(\xi_j)\, d\xi_j + \int_0^{l_j}\int_0^T w_j(\xi_j, \tau) f_j(\xi_j, \tau)\, d\xi_j d\tau \right\} + \int_0^T \delta(\tau) m(\tau)\, d\tau \to \min$$

where R-and I-dimensional vectors of dual variables $\delta(t)$ and $\mu_j(t)$, $j \in [1:J]$, respectively, are introduced together with dual functions $w_j(x_j, t)$, $j \in [1:J]$.

Let unique optimal solution of the problem with macrocontrols (1)—(3), (8)—(10), and (7) $U^{i\circ}(t) > 0$, $z_j^\circ(x_j, t)$ and of its conjugate (11) $\delta^\circ(t)$, $\mu_j^\circ(t)$, $w_j^\circ(x_j, t)$ have been determined for specified functions $\alpha_j^i(t)$. We formulate problems for individual plates, which for each $j \in [1:J]$ are defined by formulas (1)—(5), and add to functionals in (7) the terms

$$-\int_0^T \delta^\circ(\tau) [d_j(\tau) u_j(\tau) - c_j^1(\tau) z_j(0, \tau) - c_j^2(\tau) z_j(l_j, \tau)]\, d\tau$$

Let $u_j^*(t)$ $(j \in [1:J]$ be bounded measurable optimal solutions of problems for individual plates to which correspond temperature distributions $z_j^*(x_j, t)$. We determine controls $u_j^{i\circ}(t) = \alpha_j^i(t) U^{i\circ}(t)$, introduce, as in /1/, functions $\alpha_j^i(t, \rho)$, and pass to the next following iteration in conformity with that paper. We obtain

$$\alpha_j^i(t, \rho_j) = [u_j^{i\circ}(t) + \rho_j(u_j^{i*}(t) - u_j^{i\circ}(t))] \left[\sum_{j=1}^{J} (u_j^{i\circ}(t) + \rho_j(u_j^{i*}(t) - u_j^{i\circ}(t))\right]^{-1}, \quad 0 \leqslant \rho_j \leqslant 1, \quad j \in [1:J]$$

*) Ter-Krikorov A.M., A problem of optimal heating. Proc. All-Union Conf. Problems of control of processes in continuous media with separation and combustion. Kiev, June, 1979.

Substituting the fixed weighting functions in conformity with the above relation into the input problem (1) — (7), we obtain a problem with macrocontrols whose functional optimum value $g^\circ(\rho_j)$ is a function of parameter ρ_j. We have the problem of maximizing $g^\circ(\rho_j)$ for $\rho_j \in [0, 1]$. If the maximum is attained in this problem for some ρ_j° $(j \in [1:J])$, the weight functions for the next step of the iteration process is calculated using the above relation in which ρ_j° are substituted for ρ_j.

Thus at each iteration step a problem with macrocontrols containing I variables related to controls, as well as J problems for individual plates with I control variables. Finally we have the problem of maximization with J parameters ρ_j.

Let us formulate th optimality criterion for the input problem of the admissible intermediate solution $u_j^\circ(t), z_j^\circ(x_j, t)$. The respective condition consists of satisfying the equalities

$$\sum_{j=1}^{J}\left\{-\int_0^{l_j}[z_j^*(\xi_j, T)-e_j(\xi_j)]^2 d\xi_j-\int_0^T \delta^\circ(\tau)[d_j(\tau)u_j^*(\tau)-c_j^1(\tau)z_j^*(0,\tau)-c_j^2(\tau)z_j^*(l_j,\tau)]\,d\tau+\right.$$
$$\left.\int_0^{l_j}[z_j^\circ(\xi_j, t)-e_j(\xi_j)]^2 d\xi_j\right\}+\int_0^T \delta^2(\tau)\,m(\tau)\,d\tau=0 \tag{12}$$

This condition is derived as follows. Let $\nu_j^*(t)$, $v_j^*(x_j, t)$ $(j\in[1:J])$ be the optimal solutions of dual problems for each plate. Then the set $\delta^\circ(t)$, $\nu_j^*(t)$, $v_j^*(x_j, t)$ is an admissible solution of the reciprocal of the input problem. The criterion of optimality of solution $u_j^\circ(t)$, $z_j^\circ(x_j, t)$ admissible for (1) — (7) is the equality of functionals of the pair of conjugate problems

$$\sum_{j=1}^{J}\Omega_j+\int_0^T\delta^\circ(\tau)\,m(\tau)\,d\tau=-\sum_{j=1}^{J}\int_0^{l_j}[z_j^\circ(\xi_j, T)-e_j(\xi_j)]^2 d\xi_j$$

$$\Omega_j=\int_0^T\nu_j^*(\tau)\,a_j(\tau)\,d\tau+\int_0^{l_j}v_j^*(\xi_j,0)\,\varphi_j(\xi_j)\,d\xi_j+\int_0^{l_j}\int_0^T v_j^*(\xi_j,\tau)f_j(\xi_j,\tau)\,d\xi_j d\tau$$

Equality of the optimal values of functionals of pairs of conjugate problems for individual plates yields

$$\Omega_j=\int_0^{l_j}[z_j^*(\xi_j, T)-e_j(\xi_j)]^2 d\xi_j-\int_0^T\delta^\circ(\tau)[d_j(\tau)u_j^*(\tau)-c_j^1 z_j^*(0,\tau)-c_j^2(\tau)z_j^*(l_j,\tau)]\,d\tau$$

The last two relations imply (12). When solution $u_j^\circ(t)$, $z_j^\circ(x_j, t)$ $(j\in[1:J])$ is not optimal for the input problem, (12) is satisfied as a "strictly greater" inequality.

Monotonicity with respect to the functional of the iterative method is established in conformity with the scheme in /1/. We set $\rho_j=\rho$ $(j\in[1:J])$ and determine the derivative with respect to ρ at point $\rho=0$ of the optimum value of functions $g^\circ(\rho)$ of the problem with macrocontrols. For this we differentiate the Lagrange functional taking into account the formula for $\partial\alpha_j^i(t,0)/\partial\rho$ and the dependence of $z_j^\circ(x_j, t)$ on ρ. After a number of transformations with allowance for the penultimate constraints in (11) which by virtue of $U^{i\circ}(t)>0$ are satisfied as strictly valid equalities, we finally obtain

$$(g^\circ(0))'=\sum_{j=1}^{J}\int_0^T[\beta_j^1 w_j^\circ(0,\tau)\,b_j^1(\tau)+\beta_j^2 w_j^\circ(l_j,\tau)\,b_j^2(\tau)-\delta^\circ(\tau)\,d_j(\tau)-\mu_j^\circ(\tau)]\,u^*(\tau)\,d\tau \tag{13}$$

Then by integrating by parts we transform the equality

$$\sum_{j=1}^{J}\int_0^{l_j}\int_0^T[\partial w_j^\circ(\xi_j\tau)/\partial\tau+\partial^2 w_j(\xi_j,\tau)/\partial\xi_j^2]\times[z_j^*(\xi_j,\tau)-z_j^\circ(\xi_j,\tau)]\,d\xi_j\,d\tau=0$$

and finally obtain

$$\sum_{j=1}^{J}\left\{-\int_0^{l_j}2[z_j^\circ(\xi_j,T)-e_j(\xi_j)][z_j^*(\xi_j,T)-z_j^\circ(\xi_j,T)]\,d\xi_j-\right.$$
$$\int_0^T[(\beta_j^1 w_j^\circ(0,\tau)\,b_j^1(\tau)+\beta_j^2 w_j^\circ(l_j,\tau)\,b_j^2(\tau))(u_j^*(\tau)-u_j^\circ(\tau))+$$
$$\left.\delta^\circ(\tau)(c_j^1(\tau)(z_j^*(0,\tau)-z_j^\circ(0,\tau)+c_j^2(\tau)(z_j^*(l_j,\tau)-z_j^\circ(l_j,\tau))]\,d\tau\right\} \tag{14}$$

Adding to the right-hand side of (13) the expression in (14) together the following terms:

$$\int_0^T \left\{\delta^\circ(\tau)\left[m(\tau) - \sum_{j=1}^J (d_j(\tau)u_j^\circ(\tau) - c_j^1(\tau)z_j^\circ(0,\tau) - c_j^2(\tau)z_j^\circ(l_j,\tau))\right]\right\}d\tau,$$

$$\sum_{j=1}^J \int_0^T [\delta^\circ(\tau)d_j(\tau) - \beta_j^1 w_j(0,\tau)b_j^1(\tau) - \beta_j^2 w_j(l_j,\tau) + \mu_j^\circ(\tau)]u_j^\circ(\tau)\,d\tau, \quad \sum_{j=1}^J \int_0^T (a_j(\tau) - u_j^\circ(\tau))\mu_j^\circ(\tau)\,d\tau$$

which by virtue of conditions of supplementary nonrigidity are zero, and finally obtain

$$(g^\circ(0))' = \pi_1 + \pi_2 + \pi_3, \quad \pi_2 = \sum_{j=1}^J \int_0^T \mu^\circ(\tau)(a_j(\tau) - u_j^*(\tau))\,d\tau$$

$$\pi_3 = \sum_{j=1}^J \int_0^{l_j} \{[z_j^*(\xi_j T) - e_j(\xi_j)]^2 - 2[z_j^\circ(\xi_j,T) - e_j(\xi_j)] \times [z_j^*(\xi_j,T) - z_j^\circ(\xi_j,T)] - [z_j^\circ(\xi_j,T) - e_j(\xi_j)]^2\}\,d\xi_j$$

where $\pi_1 > 0$ is the left-hand side of (12); $\pi_2 \geqslant 0$ by virtue of (5), and $\pi_3 \geqslant 0$ by virtue of convexity of functional (7).

Thus $(g^\circ(0))' > 0$ which implies the monotonicity of the iterative method of expansion with respect to the functional.

The preceding investigation relates to the problem of control of plate heating defined by heat conduction equations that are one-dimensional with respect to space variables with linear constraints and a quadratic functional. However, using the formalism of /2/, it is possible to extend the decomposition method construction to more general systems with distributed parameters of the parabolic type in which appear multidimensional space variables and, also, convex constraints and a functional.

The example described below admits analytic investigation of all constructions of the decomposition method. Consider the linear problem

$$\begin{aligned}
&\partial z_1(x,t)/\partial t - \partial^2 z_1(x,t)/\partial x^2 = 0, \quad \partial z_2(x,t)/\partial t - \partial^2 z_2(x,t)/\partial x^2 = 0 \\
&z_1(x,0) = 0, \quad z_2(x,0) = 0, \quad x \in [0,l] \\
&\partial z_1(0,t)/\partial x = -[u_1(t) - z_1(0,t)], \quad \partial z_2(0,t)/\partial x = -[u_2(t) - z_2(0,t)] \\
&\partial z_1(l,t)/\partial x = [u_1(t) - z_1(l,t)], \quad \partial z_2(l,t)/\partial x = [u_2(t) - z_2(l,t)] \\
&0 \leqslant u_1(t) \leqslant n_1, \quad 0 \leqslant u_2(t) \leqslant n_2; \quad u_1(t) + u_2(t) \leqslant v \\
&I = \int_0^l [\gamma_1 z_1(\xi,T) + \gamma_2 z_2(\xi,t)]\,d\xi \to \max
\end{aligned} \tag{15}$$

where constants $n_1, n_2, v, \gamma_1, \gamma_2$ are positive, and $\gamma_2 > \gamma_1$ is set for definiteness. The problem conjugate of (15) is of the form

$$\begin{aligned}
&-\partial y_1(x,t)/\partial t - \partial^2 y_1(x,t)/\partial x^2 = 0, \quad -\partial y_2(x,t)/\partial t - \partial^2 y_2(x,t)/\partial x^2 = 0 \\
&y_1(x,t) = \gamma_1, \quad y_2(x,t) = \gamma_2, \quad x \in [0,l] \\
&\partial y_1(0,t)/\partial x = y_1(0,t), \quad \partial y_2(0,t)/\partial x = y_2(0,t) \\
&\partial y_1(l,t)/\partial x = -y_1(l,t), \quad \partial y_2(l,t)/\partial x = -y_2(l,t) \\
&\psi_1(t) - y_1(0,t) - y_1(l,t) + \chi(t) \geqslant 0, \quad \psi_2(t) - y_2(0,t) - y_2(l,t) + \chi(t) \geqslant 0 \\
&\varphi = \int_0^l [n_1\psi_1(\tau) + n_2\psi_2(\tau) + v\chi(\tau)]\,d\tau \to \min
\end{aligned} \tag{16}$$

The first four pairs of relations in (16) are integrated independently. Using the traditional technique of the Fourier method for mixed problems we obtain for optimal functions $y_1(x,t) = \gamma_1\omega(x,t)$, $y_2(x,t) = \gamma_2\omega(x,t)$, where $\omega(x,t)$ is some function symmetric about the straight line $x = l/2$ which can be represented in the form of series (not shown here). It is important to point out the existence of $T > 0$ such that for all $0 \leqslant t \leqslant T, 0 \leqslant x \leqslant l$ the inequality $\omega(x,t) \geqslant 0$ is satisfied.

To find the optimal controls of problem (15) we apply the Pontriagin principle of maximum /2/. Introducing Pontriagin's function Π we obtain the problem

$$\Pi = 2\omega(0,t)\,[\gamma_1 u_1(t) + \gamma_2 u_2(t)] \to \max, \quad u_1(t) + u_2(t) \leqslant v,\ 0 \leqslant u_1(t) \leqslant n_1, \quad 0 \leqslant u_2(t) \leqslant n_2 \tag{17}$$

Let us consider the simplest particular case of $v<n_1, v<n_2$ in which the solution of problem (17) is $u_1(t)=0, u_2(t)=v$. This after integration of the first four pairs of relations in (15), applying the indicated Fourier method, we obtain optimal distributions $z_1(x,t)=0$, $z_2(x,t)=v\Omega(x,t)$. The optimal value of the functional of problem (15) is of the form $f=v\gamma_2\Lambda$, where Λ is a constant quantity.

The problem with macrocontrols differs from (15) by the following respective relations:

$$\partial z_1(0,t)/\partial x=-[\alpha_1(t)U(t)-z_1(0,t)],\quad \partial z_2(0,t)/\partial x=-[\alpha_2(t)U(t)-z_2(0,t)]$$

$$\partial z_1(l,t)/\partial x=[\alpha_1(t)U(t)-z_1(l,t)],\quad \partial z_2(l,t)/\partial x=[\alpha_2(t)U(t)-z_2(l,t)]$$

$$\alpha_1(t)U(t)\leqslant n_1,\quad \alpha_2(t)U(t)\leqslant n_2,\quad U(t)\leqslant v,\quad U(t)\geqslant 0$$

We assume the weights $\alpha_1,\alpha_2\geqslant 0, \alpha_1+\alpha_2=1$ to be time independent. Solution of the problem with macrocontrols is then $U^0(t)=v$ and, moreover, $g^0(\alpha_1,\alpha_2)=(\alpha_1\gamma_1+\alpha_2\gamma_2)\times v\Lambda$. If the dual problem is considered in conjunction with that containing macrocontrols, we again obtain after independent integration $w_1^\circ(x,t)=\gamma_1\omega(x,t)$, $w_2^\circ(x,t)=\gamma_2\omega\times(x,t)$, $\omega(0,t)=\omega(l,t)=\omega(t)$. The duality condition for this problem yields in conformity with (11) $2(\alpha_1\gamma_1+\alpha_2\gamma_2)\omega(t)-\alpha_1\eta_1^\circ(t)-\alpha_2\eta_1^\circ(t)-\delta^\circ(t)=0$. Since $\eta_1^\circ(t)=\eta_2^\circ(t)=0$, hence $\delta^\circ(t)=2(\alpha_1\gamma_1+\alpha_2\gamma_2)\omega(t)$.

The local problem for the first plate is defined by the first five relations of (15) and the functional

$$\gamma_1\int_0^l z_1(\xi,T)\,d\xi-\int_0^T 2(\alpha_1\gamma_1+\alpha_2\gamma_2)\,\omega(\tau)\,u_1(\tau)\,d\tau\to\max$$

Using Pontriagin's principle of maximum we reduce this problem to the following:

$$2[\gamma_1-(\alpha_1\gamma_1+\alpha_2\gamma_2)]\omega(t)\,u_1(t)=\alpha_2(\gamma_1-\gamma_2)\omega(t)u_1(t)\to\max,\quad 0\leqslant u_1(t)\leqslant n_1$$

Since $\gamma_1-\gamma_2<0$, its solution is $u_1^*(t)=0$. The analogous second local problem yields

$$\alpha_1(\gamma_2-\gamma_1)\,\omega(t)u_2(t)\to\max,\ 0\leqslant u_2(t)\leqslant n_2$$

hence because of $\gamma_2-\gamma_1>0$ we have $u_2^*(t)=n_2$.

Then in conformity with constructions of the decomposition method we calculate the weighting function in terms of parameters ρ_1,ρ_2. We have

$$\alpha_1(\rho_1,\rho_2)=\alpha_1 v(1-\rho_1)/[v-\alpha_1 v\rho_1+(n_2-\alpha_2 v)\rho_2],\quad \alpha_2(\rho_1,\rho_2)=[\alpha_2 v+(n_2-\alpha_2 v)\rho_2]/[v-\alpha_1 v\rho_1+(n_2-\alpha_2 v)\rho_2]$$

Substitution of these weights into the previously determined formula for $g^\circ(\alpha_1,\alpha_2)$ results in the maximization of the following linear-fractional function.

$$g^\circ(\rho_1,\rho_2)=[\gamma_1\alpha_1 v(1-\rho_1)+\gamma_2\alpha_2 v+\gamma_2(n_2-\alpha_2 v)\rho_2]/[v-\alpha_1 v\rho_1+(n_2-\alpha_2 v)\rho_2]$$

on the unit square with respect to ρ_1,ρ_2.

It can be shown that the maximum of this function is attained when $\rho_1=1$ and any ρ_2 along segment $[0,1]$, with $u_1^\circ(\rho_1^\circ,\rho_2^\circ)=0$, $u_2^\circ(\rho_1^\circ,\rho_2^\circ)=v$. Thus the solution of problem (15) is obtained in one step.

Let us check the fulfilment of the optimality criterion (12) in the considered here case. We have

$$\pi_1=\gamma_2\int_0^l z_2^*(\xi,T)\,d\xi+\int_0^T v\delta^\circ(\tau)\,d\tau-\int_0^T(u_1^*+u_2^*)\,\delta^\circ(\tau)\,d\tau-\gamma_1\int_0^T z_1^\circ(\xi,T)\,d\xi-\gamma_2\int_0^l z_2^\circ(\xi,T)\,d\xi=\gamma_2 n_2\Lambda+$$

$$(v-n_2)\int_0^T\delta^\circ(\tau)\,d\tau-\alpha_1\gamma_1 v\Lambda-\alpha_2\gamma_2 v\Lambda$$

The constant Λ is determined using the equality of optimal values of functionals of conjugate pair of problems with macrocontrols

$$\int_0^T v\delta^\circ(\tau)\,d\tau=v(\alpha_1\gamma_1+\alpha_2\gamma_2)\Lambda$$

Taking into account the last equality we obtain $\pi_1=n_2\alpha_1(\gamma_2-\gamma_1)\Lambda$, and, consequently, when $\alpha_1=0$ we have $\pi_1=0$.

Let us consider the particular case of $v<n_1, n_2<v$, in which, as previously, we have the optimal solution of problem (15), $u_1(t)=v-n_2, u_2(t)=n_2$. Solution of the conjugate problem of

the problem with macrocontrols depends on the relation between the quantities v and n_2/α_2, where α_2 is the initial constant weighting factor. Let the following inequality be satisfied:

$$\alpha_2 < n_2 / v \tag{18}$$

Then, as before, we obtain $U^\circ(t) = v$, $\eta_1^\circ(t) = \eta_2^\circ(t) = 0$, $\delta^\circ(t) = 2(\alpha_1\gamma_1 + \alpha_2\gamma_2)\,\omega(t)$ and the local problems yield $u_1^*(t) = 0$, $u_2^*(t) = n_2$. We have the problem of maximizing the same function $g^\circ(\rho_1, \rho_2)$, as in the previous case, with respect to ρ_1, ρ_2 in the unit square, but when $\rho_1^\circ = 1$ condition (18) is violated. We consider α_2 as a function of ρ_1 of the form $\alpha_2(\rho_1) = \alpha_2 / (1 - \alpha_1\rho_1)$ when $\rho_2 = 0$, and seek point ρ_1^* at which condition (18) is not satisfied. We have the equation $\alpha_2(\rho_1) = n_2/v$ from which we obtain $\rho_1^* = (n_2 - \alpha_2 v) / n_2\alpha_2$, $\rho_1^* \in [0, 1]$. It is important to note that point $(\rho_1^*, 0)$ defines the optimal control $u_1^\circ = \alpha_1(\rho_1^*, 0)\, U^\circ$, $u_2^\circ = \alpha_2(\rho_1^*, 0) U^\circ$ and is, also, the optimal solution of problem (18). Other particular cases of this example are similarly investigated.

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Translated by J.J.D.